Interacting Conceptual Spaces

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Semantics Spaces at the Intersection of Physics, NLP, and Cognitive Science 2016

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- Conceptual spaces theory provides a way of representing structured concepts
- Categorical compositional models provide a successful model of natural language

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3. Interpret the compositional structure of the grammar category in the semantics category via a functor preserving the necessary structure.
4. Bingo! This functor maps type reductions in the grammar category onto algorithms for composing meanings in the semantics category.

Pregroup Grammar

- Pregroup: partially ordered monoid with left and right adjoints such that:

$$p^l \cdot p \leq 1 \leq p \cdot p^l \quad \text{and} \quad p \cdot p^r \leq 1 \leq p^r \cdot p \qquad (1)$$

- We construct a grammar from the alphabet $\{n, s\}$ - n for noun, s for sentence.
- Composites built by concatenation: adjective $\mapsto nn^l$, transitive verb $\mapsto n^r s n^l$.
- Composition in accordance with equation 1:

$$n\ (n^r s n^l)\ n \leq 1 \cdot s n^l n \leq 1 \cdot s \cdot 1 \leq s$$

Conceptual Spaces

- A framework for representing information at the conceptual level.
- Contrasted with symbolic and associationist approaches.
- Structures based on quality dimensions such as weight, height, hue, and so on.
- *Concepts* are roughly interpreted as convex subsets of a conceptual space.
- Conceptual spaces have internal structure based on domains.

Convex Algebras

- Notation. For a set X we write $\sum_i p_i|x_i\rangle$ for a finite formal convex sum of elements of X, where $p_i \in \mathbb{R}^{\geq 0}$ and $\sum_i p_i = 1$.
- A **convex algebra** is a set A with a **mixing operation** α satisfying:

$$\alpha(|a\rangle) = a$$

$$\alpha(\sum_{i,j} p_i q_{i,j}|a_{i,j}\rangle) = \alpha(\sum_i p_i|\alpha(\sum_j q_{i,j}|a_{i,j}\rangle)\rangle)$$

- Examples: Real or complex vector spaces, simplices, semilattices, trees
- A **convex relation** between convex algebras (A, α) and (B, β) is a relation that commutes with forming convex combinations, i.e.

$$(\forall i. R(a_i, b_i)) \Rightarrow R(\sum_i p_i a_i, \sum_i p_i b_i)$$

A Category for Conceptual Spaces

- We define the category **ConvexRel** as having convex algebras as objects and convex relations as morphisms, with composition and identities as for ordinary binary relations.
- Then **ConvexRel** has the necessary categorical structure for categorical compositional semantics
- Unit is a singleton set $\{*\}$
- Given a pair of convex algebras (A, α) and (B, β) we can form a new convex algebra on the cartesian product $A \times B$, denoted $(A, \alpha) \otimes (B, \beta)$, with mixing operation $\sum_i p_i |(a_i, b_i)\rangle \mapsto (\sum_i p_i a_i, \sum_i p_i b_i)$.
- The cap $\eta_{(A,\alpha)} : I \to (A, \alpha) \otimes (A, \alpha)$ is given by $\{(*, (a, a)) \mid a \in A\}$
- The cup $\epsilon_{(A,\alpha)} : (A, \alpha) \otimes (A, \alpha) \to I$ is its converse.

- We define a **conceptual space** to be an object of **ConvexRel**.
- We require an noun space N and a sentence space S
- A noun is a convex subset of N, and a sentence is a convex subset of S
- The noun space N will be organised into domains such as colour, texture, and so on.

Food and Drink - the Noun Space

We define a property $p_{property}$ to be a convex subset of a domain, and specify the following examples:

$$p_{yellow} = [45, 75] \times [0.5, 1] \times [0, 1], \;\; p_{green} = [75, 135] \times [0.5, 1] \times [0, 1]$$

$$p_{brown} = [0, 45] \times [0.8, 1] \times [0.2, 0.4]$$

$$p_{sweet} = \{\vec{t} | t_{sweet} \geq t_l \text{ for } l \neq sweet\}$$

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Some nouns:

$$banana = [60, 95] \times [0.75, 1] \times [0.25, 1] \times Conv(p_{sweet} \cup p_{bitter}) \times [0.2, 0.5]$$

$$apple = [0, 105] \times [0.75, 1] \times [0.5, 1] \times Conv(p_{sweet} \cup p_{sour}) \times [0.5, 0.8]$$

$$beer = [40, 50] \times [0.85, 1] \times [0.1, 0.7] \times Conv(p_{sweet} \cup p_{sour} \cup p_{bitter}) \times [0, 0.01]$$

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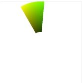
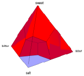
banana =  × 

× 

apple =  × 

× 

beer =  × 

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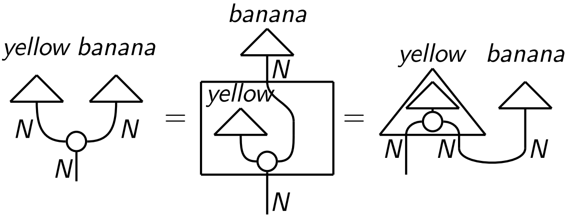
Food and Drink - the Sentence Space

- Simple example. Events are either positive or negative, surprising or unsurprising.
- Sentence space of pairs. First element states whether sentence is positive (1) or negative (0), and the second element states whether sentence is surprising (1) or unsurprising (0).
- Convex algebra is that of a join semilattice induced by elementwise max
- Sentence meanings are convex subsets of the space, for example singletons, or larger subsets such as $\textit{negative} = \{(0,\ 1), (0,\ 0)\}$.

Adjectives

- Pregroups: nn^l. **ConvexRel**: $N \otimes N$
- $yellow_{adj} = \{(\overrightarrow{x}, \overrightarrow{x}) | x_{colour} \in p_{yellow}\}$



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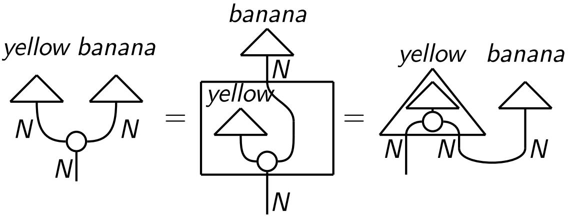
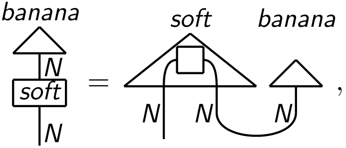


- $soft_{adj} = \{(\overrightarrow{x}, \overrightarrow{x}) | \overrightarrow{x} \in banana \text{ and } x_{texture} \leq 0.35 \text{ or } \overrightarrow{x} \in apples \text{ and } x_{texture} \leq 0.6\}$



Verbs

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$$
\begin{aligned}
taste = {} & (green\ banana \times \{(0,0)\} \times bitter) \\
& \cup (green\ banana \times \{(1,1)\} \times sweet) \\
& \cup (yellow\ banana \times \{(1,0)\} \times sweet) \\
& \cup (beer \times \{(0,1)\} \times sweet) \cup (beer \times \{(1,0)\} \times bitter)
\end{aligned}
$$

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- Sweet bananas are good:

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\textit{bananas taste sweet} &= (\epsilon_N \times 1_S \times \epsilon_N)(\textit{bananas} \times \textit{taste} \times \textit{sweet}) \\
&= (\epsilon_N \times 1_S)(\textit{banana} \times (\textit{green banana} \times \{(1,1)\} \\
&\quad \cup \textit{yellow banana} \times \{(1,0)\}) \\
&= \{(1,1),(1,0)\} = \textit{positive}
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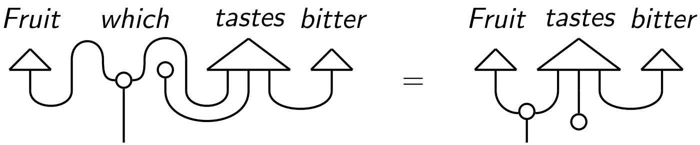
- Sweet beer is not so good:

$$\begin{aligned} \textit{beer tastes sweet} &= (\epsilon_N \times 1_S \times \epsilon_N)(\textit{beer} \times \textit{taste} \times \textit{sweet}) \\ &= \{(0,1)\} = \textit{negative and surprising} \end{aligned}$$

Relative Pronouns



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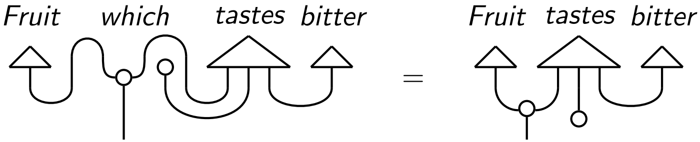

- In our example, *Fruit which tastes bitter* = *green banana*

Summary

- We have applied the categorical compositional scheme to conceptual spaces.
- We developed the category **ConvexRel**, describing convex algebras and convex relations, to model conceptual spaces.
- This allows us to represent different parts of speech and to represent sentences.
- We have demonstrated this model with a number of toy examples.
- Further work:
  - Additional structure needed: distance measures, convergence, fixed points
  - Betweenness relations
  - Negation and other logical connectives